Local Dynamics of Contracting Germs in Dimension 2

§1 (Discrete) Local dynamical systems

Let X be a complex (possibly singular) manifold of dimension d, and $f: X \to X$ a holomorphic (or meromorphic) map.

We want to study the behavior of the iterates $f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$ of f in a neighborhood of a fixed point x_0.

Assume x_0 is a smooth point. Then this boils down to studying the iterates of holomorphic germs $f: (\mathbb{C}^d, 0) \circlearrowleft$.

* Rem: f^n is not necessarily defined on a nbhd of 0.

A classical strategy consists in looking for special coordinates so that the expansion of f in formal power series is simpler.

Def: Let $f, \tilde{f}: (\mathbb{C}^d, 0) \circlearrowleft$ be two holomorphic germs.

We say that f and $\tilde{f}$ are
- holomorphically (or analytically)
- topologically
- formally

conjugate

if $\exists \phi: (\mathbb{C}^d, 0) \circlearrowleft$
- (local) biholomorphism
- (local) homeomorphism
- formal diffeomorphism

s.t. $\Phi \circ f = \tilde{f} \circ \Phi$

$$\begin{array}{ccc} (\mathbb{C}^d, 0) & \xrightarrow{\tilde{f}} & (\mathbb{C}^d, 0) \\ \simeq \uparrow \Phi & & \simeq \uparrow \Phi \\ (\mathbb{C}^d, 0) & \xrightarrow{f} & (\mathbb{C}^d, 0) \end{array}$$

We denote $f \underset{\substack{(hol)\\(top)\\(for)}}{\simeq} \tilde{f}$.

A (hol/top/for) conjugacy invariant for a family $\mathcal{A}$ of germs is a map $\mathcal{I}: \mathcal{A} \to S$ (S some set) so that $f \simeq \tilde{f} \Rightarrow \mathcal{I}(f) = \mathcal{I}(\tilde{f})$.

The invariant is complete if the opposite holds.

A normal family $\mathcal{F}$ of $\mathcal{A}$ (for hol/top/for conjugacy) is a family of germs so that $\forall f \in \mathcal{A}\ \exists \tilde{f} \in \mathcal{F}$ s.t. $f \simeq \tilde{f}$.

Rem: like this, $\mathcal{F} = \mathcal{A}$ is a normal family. The goal is to get $\mathcal{F}$ as small as possible, ideally so that $\#\{\tilde{f} \in \mathcal{F} \mid \tilde{f} \simeq f\} < +\infty$ for all $f \in \mathcal{A}$.

§2 1-dimensional case

It turns out that the situation varies strongly according to the multiplier of f.

Def: Let $f : (\mathbb{C}, 0) \circlearrowleft$ be a holomorphic germ. Its multiplier is $\lambda = f'(0)$.

Rem: The multiplier λ is a formal (hence holomorphic) invariant.

sign $\log|\lambda|$ is a topological invariant $\in \{-\infty, -1, 0, 1\}$.

If $|\lambda| \neq 1$, the study is rather easy:

$\lambda = 0$	superattracting	$f \overset{hol}{\simeq} z \mapsto z^p \quad p \geq 2$
$0 < \lambda < 1$	attracting	$f \overset{hol}{\simeq} z \mapsto \lambda z$
$\lambda > 1$	repelling	f "
$\lambda = 1$	indifferent	hard.

contracting germs : $|\lambda| < 1$

The indifferent case behave quite differently whether λ is or isn't a root of unity:

If $\lambda^q = 1$: up to taking f^q, we may assume $\lambda = 1$d. (tangent to the identity) parabolic case.

Then: $f = \mathrm{id}$ or $\exists\, r>0$ s.t. $f(z) = z(1 + a z^r + o(z^r))$, $a \neq 0$.

$r+1 = \mathrm{ord}(f - \mathrm{id}) \geq 2$ is the multiplicity of the fixed point

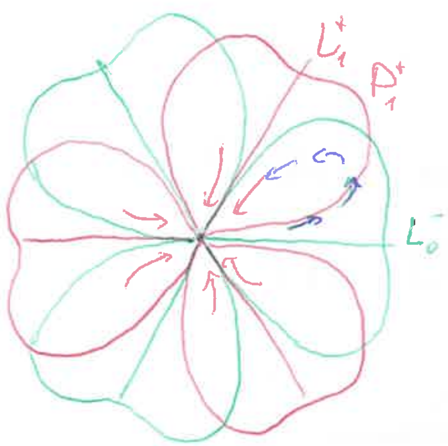
Then: $f \underset{\mathrm{hol}}{\simeq} z(1 + z^r + \beta z^{2r})$. $\beta = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - f(z)}\, dz \in \mathbb{C}$ is the index of f.

Camacho: $f \underset{\mathrm{top}}{\simeq} z(1 + z^r)$.

Rem: $z(1+z^r)$ has r invariant half lines L_j^+ where $f^n|_{L_j^+} \to 0$ (unif. on compacts)

and r invariant half lines L_j^- where $f^{-n}|_{L_j^-} \to 0$ "

Leau-Fatou: (flower theorem). f tangent to the id. of mult. $r+1$. Then: $\exists\, P_j^\pm$ $2r$ open simply connected sets,



$f^{\pm 1}(P_j^\pm) \subset P_j^\pm$, $f|_{P_j^\pm} \underset{\mathrm{hol}}{\simeq} z \mapsto z+1$.

and $\bigcup_j P_j^+ \cup P_j^-$ is a punctured neighborhood of 0.

Analytic classification: depend on the transition maps on the intersection of two consecutive petals (Écalle-Voronin).

If $\lambda = e^{2\pi i \alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ (irrational case), then:

$f \underset{\mathrm{for}}{\simeq} z \mapsto \lambda z$.

The analytical classification depends on the arithmetic properties of α.

(how well it can be approximated by rationals)

Brjuno: $\mathcal{U}_\alpha = \{ f : (\mathbb{C},0) \circlearrowleft \text{ s.t. } f'(0) = e^{2\pi i \alpha} \}$. Then
(see also Yoccoz)

$$f \underset{hol}{\sim} z \mapsto \lambda z \quad \forall f \in \mathcal{U}_\alpha \iff \sum_n \frac{\log q_{n+1}}{q_n} < +\infty, \text{ where } \frac{p_n}{q_n} \text{ are}$$

the approximants of α expanded in continued fraction. $\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}}$.

Rem: α satisfies the Brjuno condition if it is badly approximated by rationals. (α Diophantine $\Rightarrow$ α Brjuno)

§3. Contracting germs in higher dimension

Def: $f : (\mathbb{C}^d, 0) \circlearrowleft$ holomorphic germ is contracting if $\mathrm{Spec}(df_0) \subset \mathbb{D}$ the unit disc.

Prop: f is contracting $\iff$ $\exists U$ nbhd of 0 s.t. $f(U) \subset\subset U$ and $f^n(z) \to 0$ (unif. on compacts) $\forall z \in U$.

Idea proof: $\Rightarrow$ For some norm, $\Lambda < 1$ we have $\|f(z)\| \leq \Lambda \|z\|$, since $f(z) = df_0 \cdot z + O(\|z\|^2)$, and $\|df_0\| < \Lambda < 1$.
small if $\|z\| << 1$.

$\Leftarrow$ Consequence of higher dim. Schwarz lemma.

Rem: This caracterization can be used as a definition of contracting germ for $f : (X, x_0) \circlearrowleft$, (X, x_0) singular.

Other cases:

- $f:(\mathbb{C}^d,0)\circlearrowleft$ is superattracting if $\mathrm{Spec}(df_0)=\{0\} \Leftrightarrow df_0^d=0$.
  (special case of contracting.)
- ~~...~~
  - repelling if $\mathrm{Spec}(df_0)\subseteq \mathbb{C}\setminus\overline{\mathbb{D}}$.
  - hyperbolic if $\mathrm{Spec}(df_0)\cap\partial\overline{\mathbb{D}}=\emptyset$.
    (and not contracting, nor repelling),
  - tangent to the identity: $df_0=\mathrm{Id}$

Resonances.

Consider the invertible case: $f:(\mathbb{C}^d,0)\circlearrowleft$, $\mathrm{Spec}(df_0)\not\ni 0$.

The linear part L of f is invertible. Up to a linear change of coordinates, we may assume L is in Jordan normal form.

If $\lambda^1\dots\lambda^d$ are the eigenvalues of df_0, ordered so that $|\lambda^1|\geq\dots\geq|\lambda^d|>0$,

then: $f(x)=\left(\lambda^1x^1;\ \lambda^2x^2+\varepsilon^2(x^1);\dots,\ \lambda^dx^d+\varepsilon^d(x^1,\dots,x^{d-1})\right)+\underset{\substack{\uparrow\\ \text{maximal ideal of } \mathcal{O}_{\mathbb{C}^d,0}}}{\mathfrak{m}^2\times\dots\times\mathfrak{m}^2}$

In particular, the linear part is in triangular form.

To deal with this situation, it is natural to introduce a total order on multi-indices $\underline{n}=(n_1,\dots,n_d)\in\mathbb{N}^d$:

$\leq$ will denote the lexicographic order on $(\underset{\substack{\shortparallel\\ n_1+\dots+n_d}}{|\underline{n}|}, n_1,\dots,n_d)$.

In particular $\underset{\substack{\shortparallel\\ \underline{e}^1}}{(1,0,\dots,0)}>\underset{\substack{\shortparallel\\ \underline{e}^2}}{(0,1,0,\dots,0)}>\dots>\underset{\substack{\shortparallel\\ \underline{e}^d}}{(0,0,\dots,0,1)}$

Then the k-th coordinate $x^k\circ f$ of f is of the form: λ^kx^k+ h.o.t.,

higher order terms w.r.t. the total order $\leq$:

~~Re~~
$f(x)=(\lambda^1x^1+\varepsilon^1(x);\ \dots\ ;\lambda^dx^d+\varepsilon^d(x))$

Theorem (Formal Poincaré-Dulac). Let $f:(\mathbb{C}^d,0)\circlearrowleft$ be an invertible germ. Then f is formally conjugate to a germ $\tilde{f}$ of the form:
$\tilde{f}(x) = (\lambda^1 x^1 + \tilde{\varepsilon}^1(x), \dots, \lambda^d x^d + \tilde{\varepsilon}^d(x))$, where
$\tilde{\varepsilon}^k$ contain only ~~k~~- resonant monomials (for the k-th coordinate).

Def: A monomial $x^{\underline{n}} = (x^1)^{n_1}\cdots(x^d)^{n_d}$ $(\underline{n}\in\mathbb{N}^d)$ is resonant (for the k-th coordinate) if $\underline{n} > \underline{e}^k$ and $\lambda^k = \lambda^{\underline{n}} := (\lambda^1)^{n_1}\cdots(\lambda^d)^{n_d}$.

Contracting case:

Prop: If f is contracting (i.e., $|\lambda^1|<1$) then the number of resonant monomials is finite.

Proof: $\lambda^k = \lambda^{\underline{n}} \Rightarrow |\lambda^k| = \prod_{j=1}^d |\lambda^j|^{n_j} \Rightarrow -\log|\lambda^k| = -\sum_{j=1}^d n_j \log|\lambda^j| \geq |\underline{n}|\cdot(-\log|\lambda^1|)$

$\Rightarrow |\underline{n}| \leq \dfrac{-\log|\lambda^k|}{-\log|\lambda^1|}$. Only finitely many $\underline{n}$ have bounded $|\underline{n}|$. $\square$

Example: Contracting ~~invertible~~ attracting germs in dimension $d=2$:

The only possible resonance is given by $\lambda^2 = (\lambda^1)^u$ for some $u\in\mathbb{N}^*$, and $f \underset{\mathrm{for}}{\simeq} (x^1,x^2)\mapsto(\lambda^1 x^1;(\lambda^1)^u x^2 + \varepsilon(x^1)^u)$.

A concise statement would be: Normal forms are: $\tilde{f}(x,y) = (\lambda x;\mu y + \varepsilon x^u)$, with $1>|\lambda|\geq|\mu|>0$, and $(\lambda^u-\mu)\varepsilon = 0$ $(u\geq 1)$

In the attracting case, a $\tilde{f}$ given by the P-D theorem is called "Poincaré-Dulac normal form".

Theorem: (Poincaré Dulac). If $f:(\mathbb{C}^d,0)\circlearrowleft$ is an attracting germ, then f is holomorphically conjugate to any of its P-D normal forms.

To prove the formal Poincaré-Dulac theorem, we need some notations. ⑦

Let ϕ be a formal power series in d coordinates. We write

$$\phi = \sum_{\underline{n} \in \mathbb{N}^d} \phi_{\underline{n}} x^{\underline{n}}, \quad \text{where} \quad x^{\underline{n}} = (x^1)^{n_1} \cdots (x^d)^{n_d}.$$

We will have to compute powers of ϕ.

$$\phi^h = \Big(\sum_{\underline{n}} \phi_{\underline{n}} x^{\underline{n}} \Big)^h = \sum_{I \in \mathcal{N}(h)} \phi_I x^{|I|}, \quad \text{where:} \quad \phi_I = \phi_{\underline{i}^1} \cdots \phi_{\underline{i}^h} = \prod_{j=1}^{h} \phi_{\underline{i}^j}.$$

$$\mathcal{N}(h) = \{ I = (\underline{i}^1, \dots, \underline{i}^h) \quad \underline{i}^j \in \mathbb{N}^d \}, \quad \text{and} \quad |I| = \underline{i}^1 + \dots + \underline{i}^h \in \mathbb{N}^d.$$

More generally, if $\Phi = (\phi^1, \dots, \phi^d)$ is a d-uple of formal power series, we will want to compute $(\Phi)^{\underline{h}}$ for some $\underline{h} \in \mathbb{N}^d$.

We get:

$$\Phi^{\underline{h}} = \prod_{k=1}^{d} \Big(\sum_{\underline{n}^k} \phi^k_{\underline{n}^k} x^{\underline{n}^k} \Big)^{h_k} = \prod_{k=1}^{d} \prod_{l=1}^{h_k} \Big(\sum_{\underline{n}^{kl} \in \mathbb{N}^d} \phi^k_{\underline{n}^{kl}} \cdot x^{\underline{n}^{kl}} \Big) = \sum_{\underline{I} \in \mathcal{N}(\underline{h})} \phi_{\underline{I}}\, x^{|\underline{I}|}$$

where: $\mathcal{N}(\underline{h}) = \{ \underline{I} = (I^1, \dots, I^d), \ I^k \in \mathcal{N}(h^k) \}$,

$$\phi_{\underline{I}} = \phi^1_{I^1} \cdots \phi^d_{I^d} = \prod_{k=1}^{d} \phi^k_{I^k}; \qquad |\underline{I}| = |I^1| + \dots + |I^d| = \sum_{k=1}^{d} |I^k|$$

Rem: notice the absence of binomials: we are not regrouping factors together.

Examples: $\phi = \underset{\phi_2}{\varepsilon x^2} + \underset{\phi_0}{\mu y} \rightsquigarrow \phi^2 = \varepsilon^2 x^4 + 2\varepsilon\mu x^2 y + \mu^2 y^2$

$(x^1, x^2) = (x, y)$

$$I: \quad \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$\Phi = (\underset{\phi^1_{\binom{1}{0}}}{\lambda x} + \underset{\phi^1_{\binom{1}{2}}}{x y^2}, \underset{\phi^2_{\binom{2}{0}}}{\varepsilon x^2} + \underset{\phi^2_{\binom{0}{1}}}{\mu y}); \quad \Phi^{\binom{1}{2}} = \lambda \varepsilon^2 x^5 + 2\lambda\varepsilon\mu x^3 y + \lambda \mu^2 x y^2 + \varepsilon^2 x^5 y^2 + \dots$$

$$\left(\begin{array}{c|cc} 1 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{c|cc} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right) + \left(\begin{array}{c|cc} 1 & 0 & 2 \\ 0 & 1 & 0 \end{array}\right) \quad \left(\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right) \quad \left(\begin{array}{c|cc} 1 & 2 & ? \\ 2 & 0 & 0 \end{array}\right)$$

etc.

Proof (formal P.D.) Assume df_0 in Jordan normal form.

Write $f(x) = (f^1, \dots, f^d)$, with $f^k = \lambda^k x^k + \underbrace{\varepsilon^k(x)}_{\text{h.o.t.}} = \sum_{\underline{n}} \varepsilon^k_{\underline{n}} x^{\underline{n}}$

If we denote by $in_\varepsilon(f^k)$ the monomial term with lowest exponent (w.r.t ε), we have

$in_\varepsilon(f^k) = \lambda^k x^k$.

Similarly, write $\tilde{f} = (\tilde{f}^1, \dots, \tilde{f}^d)$, $\tilde{f}^k = \sum_{\underline{n}} \tilde{\varepsilon}^k_{\underline{n}} x^{\underline{n}}$.

Consider a tangent to the identity map $\Phi = (x^1 + \phi^1(x), \dots, x^d + \phi^d(x))$,

$\phi^k \in \mathfrak{M}^2$. $\forall k = 1 \dots d$.

Consider the conjugacy equation $\Phi \circ f = \tilde{f} \circ \Phi$.

We have:

$$\mathrm{I}^k := x^k \circ \Phi \circ f = \phi^k \circ f = \sum_{\underline{i}} \phi^k_{\underline{i}} f^{\underline{i}} = \sum_{\underline{i}} \phi^k_{\underline{i}} \cdot \sum_{\underline{J} \in N(\underline{i})} \varepsilon_{\underline{J}} x^{|\underline{J}|}$$

$$\mathrm{III}^k := x^k \circ \tilde{f} \circ \Phi = \tilde{f}^k \circ \Phi = \sum_{\underline{i}} \tilde{\varepsilon}^k_{\underline{i}} \cdot \Phi^{\underline{i}} = \sum_{\underline{i}} \tilde{\varepsilon}^k_{\underline{i}} \cdot \sum_{\underline{J} \in N(\underline{i})} \phi_{\underline{J}} x^{|\underline{J}|}$$

Set: $\mathrm{I}^k = \sum_{\underline{n}} \mathrm{I}^k_{\underline{n}} x^{\underline{n}}$ and $\mathrm{III}^k = \sum_{\underline{n}} \mathrm{III}^k_{\underline{n}} x^{\underline{n}}$.

We want to solve $\mathrm{I}^k_{\underline{n}} = \mathrm{III}^k_{\underline{n}}$ $\forall k = 1 \dots d$, $\underline{n} \in \mathbb{N}^d$, where:

$\varepsilon^k_{\underline{j}}$ are parameters; $\phi^k_{\underline{i}}$ are variables, as are $\tilde{\varepsilon}^k_{\underline{i}}$ (which we would like to put 0 for as many $(k, \underline{i})$ as possible.)

We have: $\mathrm{I}^k_{\underline{n}} = \sum_{\substack{\underline{i}, \underline{J} \in N(\underline{i}) \\ |\underline{J}| = \underline{n}}} \phi^k_{\underline{i}} \varepsilon_{\underline{J}}$ $\qquad \mathrm{III}^k_{\underline{n}} = \sum_{\substack{\underline{i}, \underline{J} \in N(\underline{i}) \\ |\underline{J}| = \underline{n}}} \tilde{\varepsilon}^k_{\underline{i}} \cdot \phi_{\underline{J}}$

We want to determine the h.o.t. of $\mathrm{I}^k_{\underline{n}}$ and $\mathrm{III}^k_{\underline{n}}$ w.r.t. the variables ($\phi^k_{\underline{i}}$ and $\tilde{\varepsilon}^k_{\underline{i}}$)

Since $\varepsilon^k_{\underline{j}} = 0 \ \forall\, \underline{j} < \underline{e}^k$, we get that $\varepsilon_{\underline{J}} = 0 \ \forall\, \underline{J} \in \mathcal{N}(\underline{0})$, $|\underline{J}| < \underline{i}$.

In particular, the highest order term of $\mathrm{II}^k_{\underline{n}}$ is given by $\underline{i} = \underline{n}$, and

$\underline{J} = (\underbrace{\underline{e}^1, \dots, \underline{e}^1}_{n_1 \text{ times}}, \dots, \underbrace{\underline{e}^d \dots \underline{e}^d}_{n_d \text{ times}})$. In this case $\varepsilon_{\underline{J}} = (\lambda^1)^{n_1} \cdots (\lambda^d)^{n_d} = \lambda^{\underline{n}}$

We get: $\mathrm{II}^k_{\underline{n}} = \phi^k_{\underline{n}} \lambda^{\underline{n}} + \text{l.o.t}(\phi)$.

For the second expression $\mathrm{III}^k_{\underline{n}}$, we need to control both $\tilde{\varepsilon}$ and ϕ.

For $\tilde{\varepsilon}$, the highest element is given by $\underline{i} = \underline{n}$, $\underline{J}$ as above, giving: $\tilde{\varepsilon}^k_{\underline{n}}$

(ϕ was tangent to the identity, i.e. $\phi_{\underline{J}} = 1$)

For ϕ, the highest order is given by taking $|\underline{i}| = 1$, and $\underline{J} = (0\dots0|\underline{n}|0\dots0)$

If $d f_0$ is diagonal, the only $\tilde{\varepsilon}^k_{\underline{i}} \neq 0$ for $|\underline{i}| = 1$ is $\underline{i} = \underline{e}^k$.

In this case we get $\tilde{\varepsilon}_{\underline{e}^k} = \lambda^k$ and the monomial $\lambda^k \phi_{\underline{n}}$.

So in this case: $\mathrm{III}^k_{\underline{n}} = \tilde{\varepsilon}^k_{\underline{n}} + \lambda^k \phi_{\underline{n}} + \text{l.o.t}(\phi, \tilde{\varepsilon})$.

In general, we need to put an order on the coordinates.

For a couple $(k, \underline{n})$ we define $w(k, \underline{n}) = |\underline{n}| + \frac{k}{d+1}$, so that $w(k-1, \underline{n}) < w(k, \underline{n})$.

Analogously, we could consider a total order $\underset{\text{lex}}{\leq}$ on couples $(k, \underline{n})$, given by the lexicographic order on $(|n|, n_1, \dots, n_d, d-k)$.

Doing this, we obtain again: $\mathrm{III}^k_{\underline{n}} = \tilde{\varepsilon}^k_n + \lambda^k \phi_{\underline{n}} + \text{l.o.t}_{\leq}(\phi, \tilde{\varepsilon})$

The equation $\mathrm{I}^k_{\underline{n}} = \mathrm{II}^k_{\underline{n}}$ becomes:

$$\phi^k_{\underline{n}}(\lambda^k - \lambda^{\underline{n}}) + \tilde{\varepsilon}^k_{\underline{n}} = lot_{\le}(\phi, \tilde{\varepsilon}). \qquad (*^k_{\underline{n}})$$

We proceed by induction on $(k, \underline{n})$:

If $\underline{n}$ is resonant for the k-th coordinate, $\exists! \tilde{\varepsilon}^k_{\underline{n}}$ (depending on the previous coefficients of ϕ and $\tilde{\varepsilon}$) so that $(*^k_{\underline{n}})$ is satisfied. (We set $\phi^k_{\underline{n}} \in \mathbb{C}$ as we want)

we set $\tilde{\varepsilon}^k_{\underline{n}} = 0$, and.

If $\underline{n}$ is not resonant for the k-th coordinate, $\exists! \phi^k_{\underline{n}}$ (depending on the previous coefficients of ϕ and $\tilde{\varepsilon}$) so that $(*^k_{\underline{n}})$ is satisfied.

Rem: notice that $(\tilde{\varepsilon}^k_n)$ are not unique; for the following reasons.

1) We only considered tangent to the identity conjugacies.

2) Every time we have a resonant monomial, we have freedom on the value $\phi^k_{\underline{n}}$. This may affect the values of $\tilde{\varepsilon}^k_{n'}$, $n' \geq n$.

Rem: In some cases we have a lot of resonant monomials.
For example if $\lambda^1 = \dots = \lambda^d = 1$, all monomials are resonant.

Corollary: let $f : (\mathbb{C}^d, 0) \circlearrowleft$ be an invertible germ, and f_{PD} a P.D normal form. Then $\forall N > 0$; $f \underset{hol}{\simeq} \tilde{f}$, with $\tilde{f} \equiv f_{PD} \mod (\mathfrak{M}^N)^d$.

Proof: it suffices to take the truncation of Φ given by the previous theorem at sufficiently high order.

Rem: For f contracting, the P.D normal form is triangular; $f_{PD} = (\lambda^1 x^1, \lambda^2 x^2 + \varepsilon^2(x^1), \dots, \lambda^d x^d + \varepsilon^d(x^1, \dots, x^{d-1}))$

Proof of Contracting P-D theorem:

Since $\#$resonant monomials $< +\infty$, by the corollary we may assume that $f = \tilde{f} + R$, with $\tilde{f}$ a P-D normal form, and R a rest $R \in (\mathfrak{M}^N)^d$ for $N \gg 0$.

Since f is contracting, $\exists\, r \ll 1$, $\exists\, \Lambda < 1$ s.t. f is defined for $\|x\|<r$
and $\|f(x)\| \leq \Lambda \|x\|$ $\forall x, \|x\|<r$. $\Rightarrow \|f^{\circ n}(x)\| \leq \Lambda^n \|x\|$ $\forall x, \|x\|<r$.

Now, pick N so that $\Lambda^N < |\lambda^d|$ (hence $< |\lambda^k|$ $\forall k=1\ldots d$).

We will show ~~this~~ by induction on $k=1\ldots d$ that we can holomorphically conjugate f to a map $f_{PD}+R$, $R=(R^1,\ldots,R^d)$, with $R^l \in \mathcal{M}^N$ $\forall l$, and $R^l \equiv 0$ for $1 \leq l \leq k$.

The basis for the induction is $k=0$, that we have as a starting point.

Assume: $f = f_{PD} + (0\ldots 0, R^k, R^{k+1} \ldots R^d)$.

Consider the conjugacy candidate $\Phi(x) = (x^1 \ldots x^{k-1}, x^k+\phi^k, x^{k+1} \ldots x^d)$ with $\phi^k \in \mathcal{M}^2$. The wanted normal form is $\tilde{f} = f_{PD} + (0 \ldots 0, \tilde{R}^{k+1} \ldots \tilde{R}^d)$.
The conjugacy relation $\Phi \circ f = \tilde{f} \circ \Phi$ gives:

$\forall l<k$: $x^l \circ \Phi \circ f = x^l \circ f = f_{PD}$

$x^l \circ \tilde{f} \circ \Phi = f^l_{PD} \circ \Phi = f_{PD}$ (because f_{PD} is triangular)

$l=k$: $x^k \circ \Phi \circ f = (x^k+\phi^k)\circ f = f^k_{PD} + R^k + \phi^k \circ f$.

$$x^k \circ \tilde{f} \circ \Phi = f^k_{PD} \circ \Phi = \lambda^k(x^k+\phi^k) + \underbrace{\varepsilon^k(x^{<k})}_{\text{triangularity}} = f^k_{PD} + \lambda^k \phi^k.$$

The equation $R^k + \phi^k \circ f = \lambda^k \phi^k$ has as solution: $\phi^k(x) = \sum_{n\geq 1} \frac{R^k \circ f^{\circ(n-1)}}{(\lambda^k)^n}$

Now, for $\|x\|<r$, we can estimate $|R^k(x)| \leq M\|x\|^N$ for some $M \gg 0$.
Then we get: $|\phi^k(x)| \leq \sum_{n\geq 1} |\lambda^k|^{-n} \cdot M \cdot \|f^{\circ(n-1)}(x)\|^N \leq M \cdot \sum |\lambda^k|^{-n} \cdot \Lambda^{N(n-1)} \|x\|^N$,
which converges since $\frac{\Lambda^N}{|\lambda^k|} < 1$.

Notice that $\phi^k \in \mathcal{M}^N$.

$\forall \ell > k$: $x^\ell \circ \Phi \circ f = x^\ell \circ f = f^\ell_{PD} + R^\ell$.

$$x^\ell \circ \tilde f \circ \Phi = f^\ell_{PD} \circ \Phi + \tilde R^\ell \circ \Phi$$

We set $\tilde R^\ell = \underbrace{\left(R^\ell + f^\ell_{PD} - f^\ell_{PD} \circ \Phi\right)}_{\in \mathcal{M}^U} \circ \Phi^{-1} \in \mathcal{M}^N$.

This concludes the induction step and the proof □

Related problems:

- Classification of vector fields:

$X = f^1 \frac{\partial}{\partial x^1} + \dots + f^d \frac{\partial}{\partial x^d}$: classification up to holomorphic change of coordinates. Singularity: if $X(0) = 0$.

Assume the linear part is invertible; up to linear change of coordinates;

$$X = \left(\lambda^1 x^1 + \varepsilon^1(x)\right)\frac{\partial}{\partial x^1} + \dots + \left(\lambda^d x^d + \varepsilon^d(x)\right)\frac{\partial}{\partial x^d}.$$

The attracting case corresponds to the "Poincaré domain" $0 \notin \mathrm{Conv.hull}(\{\lambda^k\})$.
In this case we have a similar result, where resonances are additive instead of multiplicative: $\lambda^k = \sum n_j \lambda^j$.

- Hopf surfaces / manifolds:

$f : (\mathbb{C}^d, 0) \circlearrowleft$ attracting germ; U nbhd of 0 s.t. $f(U) \subset\subset U$.

$S(f) := \bar U \setminus f(U) / \langle f \rangle$ defines a (smooth) complex manifold, called (primary) Hopf manifold. PD normal forms allow to study the moduli space and geometric properties of such manifolds.

In particular, one can use $f = f_{PD}$, and $S(f_{PD}) = \mathbb{C}^d \setminus \{0\} / \langle f_{PD} \rangle$.